Lecture 32

10.1 - Parametric Equations

Our goal for this section is to represent planar curves using one variable to describe how the x- & y-coordinates of the curve change.

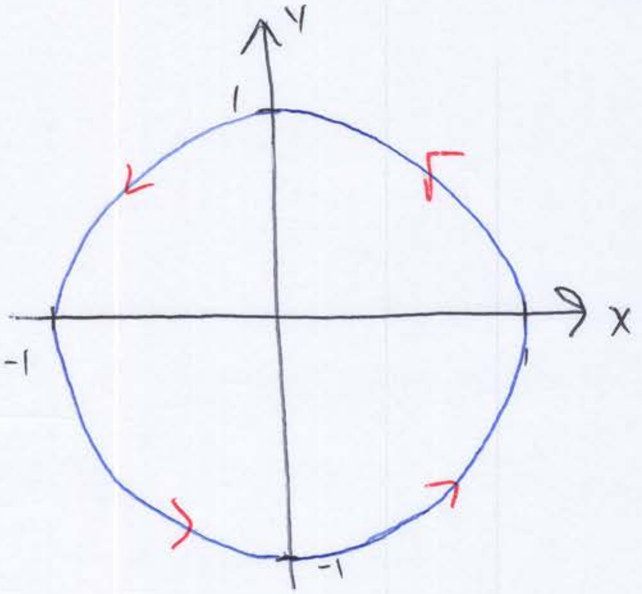
<u>Def</u>: Let $x = f(t)$ and $y = g(t)$, and let I be an interval. The set of points $(x,y) = (f(t), g(t))$ for t in I is called a <u>parametric curve</u> and $x=f(t)$ & $y=g(t)$ are called the <u>parametric equations</u> of the curve. If $I = [a,b]$, $(f(a), g(a))$ is called the <u>initial point</u> & $(f(b), g(b))$ is the terminal point.

The idea of parametric equations is that a curve is "1-dimensional" so we should be able to describe it with only one variable (locally, at least). In Calc III, you'll see this idea again, as well as for surfaces!

Ex: Sketch the parametric curve

$$x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi.$$

What is the curve?

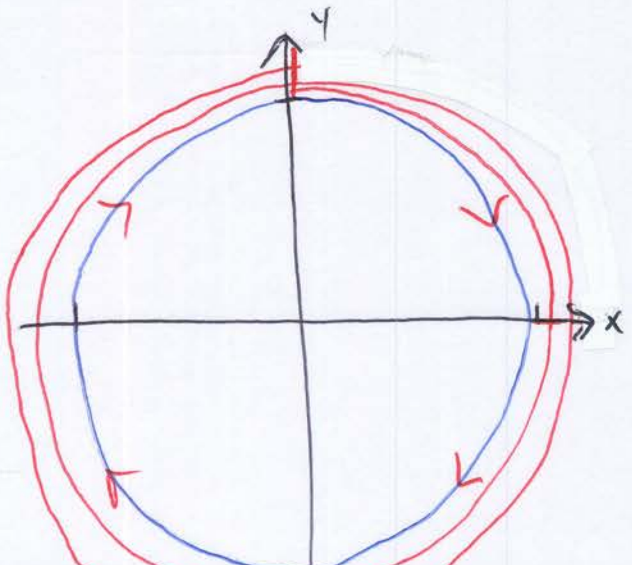


t	x	y
0	1	0
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/2$	0	1
$3\pi/4$	$-\sqrt{2}/2$	$\sqrt{2}/2$
π	-1	0
$5\pi/4$	$-\sqrt{2}/2$	$-\sqrt{2}/2$
$3\pi/2$	0	-1
$7\pi/4$	$\sqrt{2}/2$	$-\sqrt{2}/2$
2π	1	0

Ex: Sketch the parametric curve

$$x = \sin 2t,\ y = \cos 2t,\ 0 \le t \le 2\pi$$

What is the curve?



t	x	y	t
0	0	1	π
$\pi/8$	$\sqrt{2}/2$	$\sqrt{2}/2$	$9\pi/8$
$\pi/4$	1	0	$5\pi/4$
$3\pi/8$	$\sqrt{2}/2$	$-\sqrt{2}/2$	$11\pi/8$
$\pi/2$	0	-1	$3\pi/2$
$5\pi/8$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	$13\pi/8$
$3\pi/4$	-1	0	$7\pi/4$
$7\pi/8$	$-\sqrt{2}/2$	$\sqrt{2}/2$	$15\pi/8$
π	0	1	2π

Q: How is the second curve different from the first?

The second curve moves around in the opposite direction and also wraps around twice instead of once.

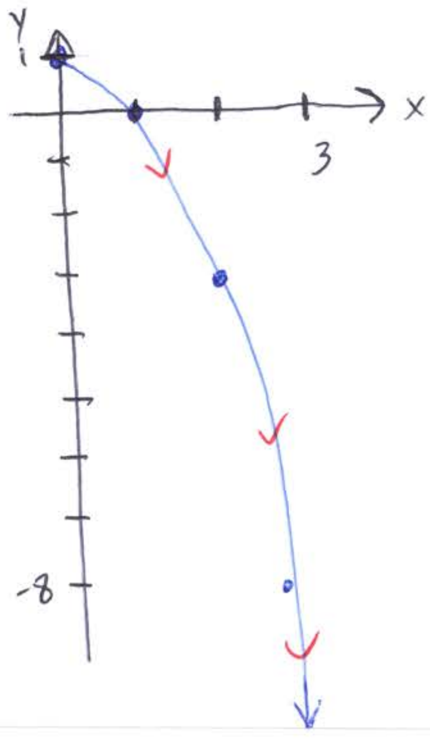
Def: The direction in which you travel along a curve is called the <u>orientation</u> of the curve. (You travel with increasing t-values.)

Ex: Sketch the parametric curve

$$x=\sqrt{t},\ y=1-t,\ 0\le t<\infty.$$

What is the orientation of the curve?

t	0	1	4	9
x	0	1	2	3
y	1	0	-3	-8



Sometimes, changing back to Cartesian coordinates is useful to identify the curve:

Ex: Describe the curve in the last example in Cartesian coordinates:

$$\left.\begin{array}{l} x=\sqrt{t} \Rightarrow x^2=t \\ y=1-t \end{array}\right\} \Rightarrow \boxed{y=1-x^2}$$

Ex: Convert the curve given by

$$x=3\sin t,\ y=2\cos t,\ \frac{\pi}{2}\le t\le \frac{9\pi}{2}$$

to Cartesian coordinates and describe the curve as t increases.

The equations fit $\frac{x^2}{9}+\frac{y^2}{4}=1$ (The 9 and 4 come in because we want to find $\sin^2 t+\cos^2 t=1$.)

$\sin t$ & $\cos t$ have period 2π & $\frac{9\pi}{2}-\frac{\pi}{2}=4\pi$, so the curve traverses the ellipse twice.

@ $t=\frac{\pi}{2}$: $(x,y)=(3,0)$ @ $t=\pi$ $(x,y)=(0,-2)$

$\Rightarrow$ Clockwise orientation.

We can also change from Cartesian to parametric.

Easiest Case: Let x or $y = t$, then solve for the other.

Ex: Give parametric equations for $f(x) = x^3$.

Let $x = t$, then $y = f(x) = t^3$

$$\boxed{x = t, y = t^3}$$

Ex: Give parametric equations for $y^2 = x^3$.

We can solve for x: $x = y^{2/3}$. So, let $y = t$
$\Rightarrow x = t^{2/3}$. $\boxed{x = t^{2/3}, y = t}$

Alternatively: $\boxed{x = t^2, y = t^3}$

Sometimes we have to rely on other known equations:

Ex: Give parametric equations describing the motion of a particle which travels around the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ counterclockwise once, starting at $(-a, 0)$.

Basing this off $\cos^2 t + \sin^2 t = 1$, x gets the $\cos t$ & y the $\sin t$ to have positive orientation. Thus $x = a\cos t$, $y = b\sin t$. To start at $(-a,0)$, we start at $t = \frac{3\pi}{2}$ $(+2n\pi)$ (for any n you want) and go 2π.

$$\boxed{x = a\cos t,\ y = b\sin t,\ \frac{3\pi}{2} \le t \le \frac{7\pi}{2}}$$